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Substructure-Based Controller Design Method for Flexible Structures

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In this paper, a decentralized procedure for designing controllers for flexible structures is presented. With decentralized control design methods, the system to be controlled is divided into subsystems for which controllers are designed. These are then combined in some way to form a controller applicable to the original system. In the proposed method, the structure to be controlled is considered as a collection of substructures. For each substructure, a subcontroller is designed with linear quadratic optimal control theory, although any method can be used. Then, a controller synthesis scheme called Substructural Controller Synthesis (SCS) is used to assemble the subcontrollers into a system controller, which is used to control the whole structure. The proposed method is closely related to component mode synthesis methods of structural dynamics and to decentralized control using overlapping subsystems. It is attractive because, unlike most decentralized methods, an explicit decomposition is generally not needed since substructure models often exist a priori. Also, an SCS controller is very adaptable because each subcontroller is designed based on an individual substructure model; therefore, the controller can be updated very economically if part of the structure changes. A plane truss example is used to illustrate the proposed method.

I. Introduction

THE problem of controlling flexible structures, also called the control/structure interaction (CSI) problem, has been an active area of research during the past decade. Typically, CSI problems combine dynamic analysis and identification of structures with application of control methods. In the conventional approach to structure control, a controller is designed based on a centralized control model that is created by reducing a larger evaluation model. Sometimes a reduced-order controller is formed to accommodate hardware constraints. The final controller is then verified with the evaluation model.

Another approach for designing controllers for flexible structures is through decentralized control. (For a discussion of decentralized control methods, see Refs. 1 and 2.) In this approach, the system to be controlled is viewed as a collection of subsystems. Controller design is carried out on the subsystem level, and the individual controllers are then applied in some manner to the complete system. The general decentralized design process is summarized in the following steps:

1) Divide the evaluation model into subsystems: For structural control problems, the subsystems are usually substructural components or groups of modes of the evaluation model.
2) Reduce the subsystem models as necessary.
3) Design controllers for the individual subsystems.
4) Combine the subsystem controllers into a controller for the complete system.
5) Create a reduced-order controller as necessary to make hardware implementation feasible.
6) Verify closed-loop stability and performance.

For structural control applications, this approach is particularly attractive because it addresses the fundamental problem of model dimensionality. Essentially, it is easier to solve a number of small problems rather than one large one. Furthermore, with an appropriate definition of subsystems, the decentralized control paradigm makes it possible to consider limitations on information transfer between sensors, controllers, and actuators. It also becomes feasible, during the controller design process, to consider the possible connection or disassembly of components.

In structural dynamics modeling, the two dominant types of subsystems are physical substructures and groups of correlated modes. Recently there has been a growing interest in using physical substructures in decentralized control.[3–7] In Ref. 3, Yousuff applied the concepts of overlapping decomposition and the inclusion principle, developed by Ikeda and Šiljak in Refs. 2, 8, and 9, to structural systems described in matrix second-order form. Later, Young introduced a method called Controlled Component Synthesis (CCS)[4] in which he interpreted the result of overlapping decomposition applied to structural dynamics equations in matrix second-order form. The components, or subsystems, used in the CCS approach are configuration-space substructure models with inertia and stiffness loading from adjacent substructures at the interface degrees of freedom. They constitute a special case of the expanded components described by Yousuff.[3] A CCS controller is created using what Young calls the interlocking control concept, which is essentially a linear quadratic regulator (LQR) (i.e., full-state feedback) with the performance index chosen such that the displacement and velocity of a specific set of nodes is minimized. In Ref. 10 the authors modified the CCS method to create a decentralized output-feedback controller.

This paper describes another substructure-based decentralized design method called Substructural Controller Synthesis (SCS), which uses substructures as the subsystems in the decentralized design of a controller and exploits the physical relationships of compatibility and equilibrium between the substructures to create a controller applicable to the complete structure. The basic idea is that controllers will be designed for individual substructures, and a global controller will be created by assembling the subcontrollers by the same process that is used to assemble the substructure models. This process is shown to be related to overlapping decompositions of generalized state-space models, also known as descriptor models.

The SCS method is attractive for a number of reasons. First, like all decentralized methods, it relieves the computational burden associated with dimensionality since controller design is carried

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out at the subsystem level where the models are of much lower order than the assembled-structure model. This can yield a substantial benefit because the solution of $N \times N$ Riccati equations used in modern control methods requires roughly N^3 operations. Next, an explicit decomposition of the complete structure is not necessary because the substructure models often exist a priori. Finally, the SCS controller is highly adaptable because each subcontroller is designed based on each substructure independently, and therefore it can be updated very economically if part of the structure changes.

This paper is organized as follows. In Sec. II, the decomposition of structural dynamics systems is discussed, and a concept called substructuring decomposition is defined. This concept is essentially an extension of the well-known substructure assembly method to generalized state-space models. In Sec. III, the SCS method is presented. Next, in Sec. IV, overlapping decomposition and the inclusion principle are defined for generalized state-space systems so that, in Sec. V, we can relate substructuring decomposition and the SCS method to them. In Sec. VI, a plane cantilever-truss numerical example is used to illustrate the proposed method. The paper concludes with Sec. VII.

II. Decomposition of Structural Dynamics Systems

To define substructuring decomposition of a structural dynamics system, consider a structure composed of two substructures that have a common interface, as shown in Fig. 1. (The procedure can be extended to more than two substructures in a straightforward manner.) The dynamics of the assembled structure (i.e., the structure as a whole) can usually be described by a second-order matrix differential equation and an output equation:

$$\begin{aligned} M\ddot{x} + D\dot{x} + Kx &= Pu + Nw \qquad x \in \Re^n,\ u \in \Re^l,\ w \in \Re^k \\ y &= C_p x + C_v \dot{x} + v \qquad y, v \in \Re^m \end{aligned} \tag{1}$$

where x is the vector of physical or generalized displacement coordinates of the structure, u is the vector of inputs, and w is the vector of disturbances; y is the output vector and v represents measurement noise. The terms M, D, and K are the mass, damping, and stiffness matrices, respectively; P is the force distribution matrix and N is the matrix that distributes the disturbances; and C_p and C_v are the displacement and velocity sensor distribution matrices, respectively.

It is assumed that the inputs and the outputs are localized. In the present context, the term localized inputs means that actuators and disturbance sources are distributed so that u_α and w_α are only applied directly to the α substructure and u_β and w_β are only applied directly to the β substructure. Localized outputs means that there are no sensors at the boundary nodes (see Fig. 1).

Similarly, the equations of motion of the two substructures can be represented by

$$\begin{aligned} M_j\ddot{x}_j + D_j\dot{x}_j + K_j x_j &= P_j u_j + N_j w_j \\ y_j &= C_{p_j} x_j + C_{v_j}\dot{x}_j + v_j \end{aligned} \qquad j = \alpha, \beta \tag{2}$$

Since the two substructures have a common interface, the displacement vectors of the substructures and the assembled structure are related. Let x_j represent the physical displacement coordinates of substructure j, and let the physical coordinates of the substructure be partitioned into two sets: interior coordinates (i set) and boundary coordinates (b set), as shown in Fig. 1. The displacement

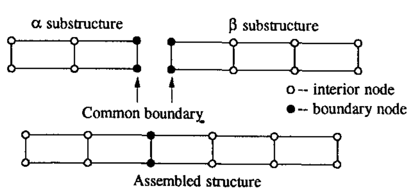

Fig. 1 Two-substructure model.

compatibility condition requires that $x_\alpha^b = x_\beta^b$. If the displacement vector of the assembled structure is represented by

$$x = \begin{Bmatrix} x_\alpha^i \\ x^b \\ x_\beta^i \end{Bmatrix}$$

where x^b is the vector of interface degrees of freedom, then the three displacement vectors x_α, x_β, and x are related by a coupling matrix T:

$$\begin{Bmatrix} x_\alpha \\ x_\beta \end{Bmatrix} \equiv \begin{Bmatrix} x_\alpha^i \\ x_\alpha^b \\ x_\beta^i \\ x_\beta^b \end{Bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix} \begin{Bmatrix} x_\alpha^i \\ x^b \\ x_\beta^i \end{Bmatrix} \equiv \begin{bmatrix} T_\alpha \\ T_\beta \end{bmatrix} x = Tx \tag{3}$$

with

$$T_\alpha = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}, \qquad T_\beta = \begin{bmatrix} 0 & 0 & I \\ 0 & I & 0 \end{bmatrix}$$

It can be readily shown (e.g., see Refs. 11 and 12) that the system matrices of the assembled structure and the system matrices of the substructures satisfy the following relations:

$$\begin{gathered} M = T^T \begin{bmatrix} M_\alpha & 0 \\ 0 & M_\beta \end{bmatrix} T, \qquad D = T^T \begin{bmatrix} D_\alpha & 0 \\ 0 & D_\beta \end{bmatrix} T \\ K = T^T \begin{bmatrix} K_\alpha & 0 \\ 0 & K_\beta \end{bmatrix} T, \qquad P = T^T \begin{bmatrix} P_\alpha & 0 \\ 0 & P_\beta \end{bmatrix} \\ N = T^T \begin{bmatrix} N_\alpha & 0 \\ 0 & N_\beta \end{bmatrix} \\ C_p = \begin{bmatrix} C_{p_\alpha} & 0 \\ 0 & C_{p_\beta} \end{bmatrix} T, \qquad C_v = \begin{bmatrix} C_{v_\alpha} & 0 \\ 0 & C_{v_\beta} \end{bmatrix} T \end{gathered} \tag{4}$$

The above relationships show an inherent property of structural dynamics systems, namely that the system matrices of the assembled structure can be obtained by assembling the system matrices of the substructures. This property is, in fact, the essence of all matrix assemblage methods (e.g., the finite element method and component mode synthesis).[12]

The above formulation is based on the matrix second-order form of the equations of motion. For controller design purposes, it is more convenient to have the equations of motion in first-order form, so let us rewrite the complete structure's system equations (1) in the following first-order form and let us call this system $\mathcal{S}$:

$$\mathcal{S}: \begin{cases} \begin{bmatrix} D & M \\ M & 0 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \ddot{x} \end{Bmatrix} = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} + \begin{bmatrix} P \\ 0 \end{bmatrix} u + \begin{bmatrix} N \\ 0 \end{bmatrix} w \\ y = [C_p \quad C_v] \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} + v \end{cases} \tag{5}$$

We can write these equations in more compact form as

$$\mathcal{S}: \begin{cases} E\dot{z} = Az + Bu + \Gamma w & z \in \Re^{2n}, \quad u \in \Re^l, \quad w \in \Re^k \\ y = Cz + v & y, v \in \Re^m \end{cases} \tag{6}$$

where z is the state vector $z = (x^T \dot{x}^T)^T$ and the meanings of u, w, y, and v are unchanged from Eqs. (1).

Similarly, the system equations of the substructures may be written in first-order form as

$$\begin{aligned} &\begin{bmatrix} D_j & M_j \\ M_j & 0 \end{bmatrix} \begin{Bmatrix} \dot{x}_j \\ \ddot{x}_j \end{Bmatrix} = \begin{bmatrix} -K_j & 0 \\ 0 & M_j \end{bmatrix} \begin{Bmatrix} x_j \\ \dot{x}_j \end{Bmatrix} \\ &\quad + \begin{bmatrix} P_j \\ 0 \end{bmatrix} u_j + \begin{bmatrix} N_j \\ 0 \end{bmatrix} w_j \\ &y_j = \begin{bmatrix} C_{p_j} & C_{v_j} \end{bmatrix} \begin{Bmatrix} x_j \\ \dot{x}_j \end{Bmatrix} + v_j \qquad j = \alpha, \beta \end{aligned} \tag{7}$$

or, using more compact notation, they may be written as

$$E_j \dot{z}_j = A_j z_j + B_j u_j + \Gamma_j w_j$$
$$y_j = C_j z_j + v_j \qquad j = \alpha, \beta \tag{8}$$

where z_j is the state vector of the jth substructure, $z_j = (x_j^T \, \dot{x}_j^T)^T$.

The individual substructure system equations can be combined into a block-diagonal system that will be called the unassembled system and labeled $\tilde{S}$:

$$\tilde{S}: \begin{cases} \tilde{E}\dot{\tilde{z}} = \tilde{A}\tilde{z} + \tilde{B}u + \tilde{\Gamma}w \\ y = \tilde{C}\tilde{z} + v \end{cases} \tag{9}$$

where

$$\tilde{E} = \begin{bmatrix} E_\alpha & \\ & E_\beta \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} A_\alpha & \\ & A_\beta \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_\alpha & \\ & B_\beta \end{bmatrix}$$

$$\tilde{\Gamma} = \begin{bmatrix} \Gamma_\alpha & \\ & \Gamma_\beta \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} C_\alpha & \\ & C_\beta \end{bmatrix}$$

and

$$\tilde{z} = \begin{bmatrix} z_\alpha^T & z_\beta^T \end{bmatrix}^T, \quad u = \begin{bmatrix} u_\alpha^T & u_\beta^T \end{bmatrix}^T, \quad w = \begin{bmatrix} w_\alpha^T & w_\beta^T \end{bmatrix}^T$$

$$y = \begin{bmatrix} y_\alpha^T & y_\beta^T \end{bmatrix}^T, \quad v = \begin{bmatrix} v_\alpha^T & v_\beta^T \end{bmatrix}^T$$

The state vector of S and the state vector of $\tilde{S}$ are related by a coupling matrix $\tilde{T}$, as follows:

$$\underset{(\tilde{z})}{\begin{Bmatrix} x_\alpha \\ \dot{x}_\alpha \\ x_\beta \\ \dot{x}_\beta \end{Bmatrix}} = \underset{(\tilde{T})}{\begin{bmatrix} T_\alpha & 0 \\ 0 & T_\alpha \\ T_\beta & 0 \\ 0 & T_\beta \end{bmatrix}} \underset{(z)}{\begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}} \tag{10}$$

This matrix is the first-order analog of the coupling matrix T introduced in Eq. (3). Physically, it simply describes the compatibility conditions that must be imposed on the interface states. The displacement compatibility condition is given in Eq. (3). The velocity compatibility condition requires that $\dot{x}_\alpha^b = \dot{x}_\beta^b$, which leads to

$$\dot{x}_\alpha = T_\alpha \dot{x}, \qquad \dot{x}_\beta = T_\beta \dot{x} \tag{11}$$

Equations (3) and (11) together show that the state vector of $\tilde{S}$ and the state vector of S are related by the coupling matrix $\tilde{T}$. In general applications, the x_α, x_β, and x vectors do not have to be described in physical coordinates. For instance, in component mode synthesis methods, the dynamics of the substructures are represented by a set of assumed modes called component modes. In this case, the column vectors in the T_α and T_β matrices would be the representation of those component modes. Even though the displacement vectors would be in generalized coordinates instead of physical coordinates, the state compatibility condition can still be described by Eq. (10).

Using this first-order coupling matrix, the system matrices of the assembled structure, S, and the system matrices of the unassembled system, $\tilde{S}$, satisfy the following relations:

$$E = \tilde{T}^T \tilde{E} \tilde{T}, \qquad A = \tilde{T}^T \tilde{A} \tilde{T}, \qquad B = \tilde{T}^T \tilde{B}$$
$$\Gamma = \tilde{T}^T \tilde{\Gamma}, \qquad C = \tilde{C} \tilde{T} \tag{12}$$

and we call $\tilde{S}$ a substructuring decomposition of S. The concept of substructuring decomposition is most intuitive when applied to structural dynamics systems; however, it can be applied to any dynamic system that can be described by generalized first-order equations like those in Eqs. (6) and (9). In the next section, a substructuring decomposition of the closed-loop dynamics equations is presented.

III. SCS Design

In this section we discuss the SCS method, a method by which controllers are designed for individual substructures and are then assembled to form a global controller applicable to a structure assembled from the substructures. For each substructure, a subcontroller is designed with linear quadratic (LQ) optimal control theory; however, any design method can be used. Then, the same coupling scheme that is employed for the plant is also used to synthesize the subcontrollers into a coupled system controller.

Let the dynamic equations of the two substructures in Fig. 1 be represented by Eq. (9) and introduce another set of outputs representing signals to be minimized by the controller,

$$y_{m_j} = H_j z_j \qquad j = \alpha, \beta \tag{13}$$

where the H_j matrices are chosen so that

$$\begin{bmatrix} H_\alpha & 0 \\ 0 & H_\beta \end{bmatrix} \tilde{T} = H \tag{14}$$

A performance index is associated with each substructure,

$$J_j = \frac{1}{2} \int_0^\infty \left[z_j^T Q_j z_j + u_j^T R_j u_j \right] \mathrm{d}t \qquad j = \alpha, \beta \tag{15}$$

where

$$Q_j = Q_{e_j} + \rho Q_{m_j} \tag{16}$$

with

$$Q_{e_j} = \begin{bmatrix} K_j & 0 \\ 0 & M_j \end{bmatrix}, \qquad Q_{m_j} = H_j^T H_j \tag{17}$$

All of the substructures must be completely controllable and observable with the given performance weightings and noise distributions. For each substructure, a linear quadratic Gaussian (LQG) design can be carried out to obtain an optimal subcontroller in the form

$$E_j \dot{q}_j = \left(A_j + B_j G_j^\circ - F_j^\circ C_j \right) q_j + F_j^\circ y_j$$
$$u_j = G_j^\circ q_j \qquad j = \alpha, \beta \tag{18}$$

where q_j is an estimate of the state vector z_j and the superscript $\circ$ denotes an optimal design for the substructure.

In order to illustrate how the concept of substructuring decomposition is used to assemble the subcontrollers, we write the controller equations in the diagonal form of $\tilde{S}$ as

$$\tilde{E}\dot{\tilde{q}} = (\tilde{A} + \tilde{B}\tilde{G}^\circ - \tilde{F}^\circ \tilde{C})\tilde{q} + \tilde{F}^\circ y$$
$$u = \tilde{G}^\circ \tilde{q} \tag{19}$$

with

$$\tilde{G}^\circ = \begin{bmatrix} G_\alpha^\circ & 0 \\ 0 & G_\beta^\circ \end{bmatrix}, \qquad \tilde{F}^\circ = \begin{bmatrix} F_\alpha^\circ & 0 \\ 0 & F_\beta^\circ \end{bmatrix}$$

The combination of Eqs. (9) and (19) gives the following closed-loop equation for $\tilde{S}$:

$$\begin{bmatrix} \tilde{E} & 0 \\ 0 & \tilde{E} \end{bmatrix} \begin{Bmatrix} \dot{\tilde{z}} \\ \dot{\tilde{q}} \end{Bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B}\tilde{G}^\circ \\ \tilde{F}^\circ \tilde{C} & \tilde{A} + \tilde{B}\tilde{G}^\circ - \tilde{F}^\circ \tilde{C} \end{bmatrix} \begin{Bmatrix} \tilde{z} \\ \tilde{q} \end{Bmatrix} + \begin{bmatrix} \tilde{\Gamma} & 0 \\ 0 & \tilde{F}^\circ \end{bmatrix} \begin{Bmatrix} w \\ v \end{Bmatrix} \tag{20}$$

A controller for the assembled structure is created by assembling the subcontrollers with the same coupling scheme that was used for assembling the substructures. The global controller equations may be written as

$$E\dot{q} = (A + BG^\oplus - F^\oplus C)q + F^\oplus y$$
$$u = G^\oplus q \tag{21}$$

with

$$F^{\oplus} = \tilde{T}^T \tilde{F}^{\circ}, \qquad G^{\oplus} = \tilde{G}^{\circ} \tilde{T} \tag{22}$$

where the superscript $\oplus$ denotes that the controller is not optimal but suboptimal. The gain matrices $F^{\oplus}$ and $G^{\oplus}$ are assembled from $\tilde{F}^{\circ}$ and $\tilde{G}^{\circ}$ with the coupling matrix $\tilde{T}$ as shown in Eq. (22). When the assembled controller is used to control the assembled structure, the following closed-loop equation is obtained:

$$\begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} \begin{Bmatrix} \dot{z} \\ \dot{q} \end{Bmatrix} = \begin{bmatrix} A & BG^{\oplus} \\ F^{\oplus}C & A + BG^{\oplus} - F^{\oplus}C \end{bmatrix} \begin{Bmatrix} z \\ q \end{Bmatrix} + \begin{bmatrix} \Gamma & 0 \\ 0 & F^{\oplus} \end{bmatrix} \begin{Bmatrix} w \\ v \end{Bmatrix} \tag{23}$$

Under this assembling scheme, the unassembled controller is a substructuring decomposition of the assembled controller and the unassembled closed-loop equation is a substructuring decomposition of the assembled closed-loop equation. The unassembled controller is optimal for the unassembled system and the assembled controller can be considered to be suboptimal for the assembled system if it yields a stable design. Closed-loop stability of a SCS design is not guaranteed; however, this is true of controllers designed with decentralized methods in general. For example, Young's CCS method[4] and the decentralized methods described in Ref. 2 also suffer from this lack of guaranteed stability. Nevertheless, these design methods have been successfully demonstrated in laboratory experiments (see Refs. 1 and 10).

From the form of Eq. (23), it is seen that the separation principle is applicable to the SCS-controlled system. The closed-loop poles of the assembled system are the union of the regulator poles (zeros of $|sE - A - BG^{\oplus}|$) and the observer poles (zeros of $|sE - A + F^{\oplus}C|$). Therefore, stability of the assembled closed-loop system can be checked by examining the locations of these two sets of eigenvalues.

It is clear that the coupling matrix $\tilde{T}$ plays the major role in the SCS method. It also plays an important role in the interpretation of the performance indices. The performance index of the unassembled system is simply the summation of the performance indices of the substructures

$$\tilde{J} = J_\alpha + J_\beta = \frac{1}{2} \int_0^\infty [\tilde{z}^T \tilde{Q} \tilde{z} + u^T \tilde{R} u]\, dt \tag{24}$$

with

$$\tilde{Q} = \begin{bmatrix} Q_\alpha & 0 \\ 0 & Q_\beta \end{bmatrix}, \qquad \tilde{R} = \begin{bmatrix} R_\alpha & 0 \\ 0 & R_\beta \end{bmatrix}$$

The state weighting matrix of the unassembled system, $\tilde{Q}$, and the state weighting matrix of the assembled system, Q, are related by $Q = \tilde{T}^T \tilde{Q} \tilde{T}$. By using the relations $\tilde{z} = \tilde{T} z$, $Q = \tilde{T}^T \tilde{Q} \tilde{T}$, and $\tilde{R} = R$, it can be shown that $\tilde{J}$, the performance index of the unassembled system, is symbolically equal to J, the performance index of the assembled system. Therefore, with the control and state weighting matrices appropriately chosen, the performance index is also substructurally decomposable. That is, the optimal control design problem for the assembled structure can be viewed as a collection of several substructural LQ problems tied together by a set of compatibility conditions.

To get a clearer idea about how the controller designs and performance indices are related, consider the following three optimization problems:

Problem 1.
Minimize

$$J = \int_0^\infty \frac{1}{2} (z^T Q z + u^T R u)\, dt$$

Subject to

$$E\dot{z} = Az + Bu$$

Problem 2.
Minimize

$$\tilde{J} = \int_0^\infty \frac{1}{2} (\tilde{z}^T \tilde{Q} \tilde{z} + u^T \tilde{R} u)\, dt$$

Subject to

$$\tilde{E}\dot{\tilde{z}} = \tilde{A}\tilde{z} + \tilde{B}u$$

Problem 3.
Minimize

$$\tilde{J} = \int_0^\infty \frac{1}{2} (\tilde{z}^T \tilde{Q} \tilde{z} + u^T \tilde{R} u)\, dt$$

Subject to

$$\tilde{E}\dot{\tilde{z}} = \tilde{A}\tilde{z} + \tilde{B}u, \qquad z_\alpha^b = z_\beta^b$$

Problem 1 is the optimal state feedback control problem for the assembled system. Problem 2 is the optimal state feedback control problem for the unassembled system. The difference between problems 3 and 2 is that problem 3 includes one more constraint condition, namely the state compatibility condition. Hence, problem 3 is the optimal state feedback control problem for the unassembled system with the compatibility condition enforced on the boundary degrees of freedom, which means that problem 3 is, in fact, exactly equivalent to problem 1. In the SCS design, it is problem 2 instead of problem 3 that is solved to obtain the subcontrollers. Therefore, in some sense, the SCS design method can be interpreted as simplifying the assembled optimal control design problem (problem 3) by ignoring some constraints (the compatibility condition).

By using the SCS method, the controller design is simplified since the controller design is carried out at the substructure level. Another potential advantage of using the SCS method is that the resulting controller is highly adaptable. For a structure with varying configuration or varying mass and stiffness properties, like some space structures, the SCS method may be especially efficient. The controller can be updated economically by simply carrying out redesign of subcontrollers associated with those substructures that have changed. On the other hand, for a controller based on a centralized design scheme, a slight change of the structure may require a full-scale redesign. In Ref. 7, there is an example to show how an SCS controller can be updated very economically when some actuators and sensors associated with some substructures have malfunctioned. This favorable decentralized feature of the SCS method is similar to that of the component mode synthesis methods with regard to model modification.

The steps of the SCS method are very straightforward and are summarized below:

1) Design a controller for each substructure.
2) Assemble the substructure models to obtain a model of the complete structure.
3) Assemble all of the subcontrollers into a global controller for the assembled structure.
4) Verify stability and performance of the closed-loop system.

IV. Overlapping Decompositions for Generalized State-Space Systems

In this section, we discuss the concepts of disjoint and overlapping decompositions and we will develop the inclusion principle (Refs. 2, 8, 9, and 13) for generalized state-space systems. The main reason for discussing decompositions of generalized systems is that we want to apply decentralized control concepts to structural systems without destroying the form of the structural matrices, as happens when standard-form state-space models are used. Generalized-form models are very useful because they are appropriate for control engineering applications, yet they retain the essential characteristics of the second-order structural system [see Eqs. (5)]. In the following section, we will show that the ideas of substructuring decomposition and the SCS approach to controller design are closely related to overlapping decomposition principles.

Generalized state-space models, also called descriptor models, are of the form

$$\Sigma: \begin{cases} E\dot{z}(t) = Az(t) + Bu(t) \qquad Ez(0) = Ez_0 \\ y(t) = Cz(t) \end{cases} \tag{25}$$

A unique $z(t)$ always exists for standard-form time-invariant state-space models (i.e., when $E = I$). However, existence and uniqueness of $z(t)$ in a generalized dynamic system depend on the E and A matrices, the input $u(t)$, and the initial conditions z_0. For simplicity, we assume that $u(t)$ is continuous at $t = 0$. Assuming that $|sE - A| \neq 0$ identically, then $|sE - A|$ is a polynomial in s with a finite number of zeros, where s is the Laplace operator. Therefore, $(sE - A)^{-1}$ exists (except at its poles) and $z(t)$ is unique. Note that neither E nor A need be invertible for $z(t)$ to be unique.

Existence of $z(t)$ may depend on $u(t)$ and z_0. When E is full rank, the system is irreducible and the initial conditions can be arbitrary. When E is not invertible, the dynamic system is a nonminimal realization, and the state space is reducible [i.e., the system contains some constraints that may involve the $u(t)$]. When the initial conditions z_0 satisfy all embedded constraints, they are satisfied for all t so $z(t)$ exists. We say that $u(t)$ and z_0 are an admissible pair if they satisfy the constraints in the system.

Regardless of the characteristics of the dynamic system, the fundamental tenet of decentralized control is the decomposition principle. It implies that certain dynamic systems made up of interacting elements can be decomposed into smaller subsystems that can be considered individually. Controllers are designed for the subsystems and are then combined in some manner to yield a controller for the overall system. Three important concepts from decentralized control theory are disjoint decompositions, overlapping decompositions, and the inclusion principle.[1,2,8,9,13]

Consider a linear time-invariant dynamic system Σ described by Eqs. (25) where $|sE - A| \neq 0$ identically, with admissible input and initial conditions so that a unique $z(t)$ exists. This system can be partitioned into ν interconnected subsystems,

$$\begin{aligned} E_r\dot{z}_r + \sum_{s=1}^{\nu} E_{rs}\dot{z}_s &= A_r z_r + B_r u_r + \sum_{s=1}^{\nu} A_{rs} z_s + \sum_{s=1}^{\nu} B_{rs} u_s \\ y_r &= C_r z_r + \sum_{s=1}^{\nu} C_{rs} z_s \qquad r = 1, \dots, \nu \end{aligned} \tag{26}$$

The decoupled subsystems are defined to be

$$E_r\dot{z}_r = A_r z_r + B_r u_r, \qquad y_r = C_r z_r \qquad r = 1, \dots, \nu \tag{27}$$

Since Eq. (27) is a disjoint decomposition,

$$z = \left[z_1^T, \dots, z_\nu^T\right]^T, \quad u = \left[u_1^T, \dots, u_\nu^T\right]^T, \quad y = \left[y_1^T, \dots, y_\nu^T\right]^T$$

This means that the decoupled subsystems are independent of each other and do not share any information (i.e., there is no dynamic spillover). In compact notation, Eqs. (26) can be written as

$$\begin{aligned} E_d\dot{z} + E_o\dot{z} &= A_d z + B_d u + A_o z + B_o u \\ y &= C_d z + C_o z \end{aligned} \tag{28}$$

and Eqs. (27) can be written as

$$E_d\dot{z} = A_d z + B_d u, \qquad y = C_d z \tag{29}$$

where the matrices E_d, A_d, B_d, and C_d are block diagonal. The decoupled subsystems are completely independent, and the $E_o\dot{z}$, $A_o z$, $B_o u$, and $C_o z$ terms represent all of the interconnections between the subsystems.

Controllers are designed for the decoupled systems but are then applied to the complete system. Although the controllers are guaranteed to stabilize the decoupled systems, there is no guarantee of stability, or even adequate performance, when the controllers are applied to the interconnected system. This is due to the effects of the interconnection terms.

The decomposition that is of primary interest in this paper is an overlapping decomposition.[2,8] Consider a system described by the following generalized state equation:

$$\begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} \begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{30}$$

The lines drawn in the matrices identify two subsystems that overlap and that share the z_2 state. To clarify the interconnection and to admit the use of standard disjoint decomposition techniques, the original system is expanded to a larger space. For example, the system described by Eq. (30) can be expanded to yield

$$\left[\begin{array}{cc|cc} E_{11} & E_{12} & 0 & E_{13} \\ E_{21} & E_{22} & 0 & E_{23} \\ \hline E_{21} & 0 & E_{22} & E_{23} \\ E_{31} & 0 & E_{32} & E_{33} \end{array}\right] \begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \hline \dot{z}_2 \\ \dot{z}_3 \end{Bmatrix} = \left[\begin{array}{cc|cc} A_{11} & A_{12} & 0 & A_{13} \\ A_{21} & A_{22} & 0 & A_{23} \\ \hline A_{21} & 0 & A_{22} & A_{23} \\ A_{31} & 0 & A_{32} & A_{33} \end{array}\right] \times \begin{Bmatrix} z_1 \\ z_2 \\ \hline z_2 \\ z_3 \end{Bmatrix} + \left[\begin{array}{c|c} B_{11} & B_{12} \\ B_{21} & B_{22} \\ \hline B_{21} & B_{22} \\ B_{31} & B_{32} \end{array}\right] \begin{Bmatrix} u_1 \\ \hline u_2 \end{Bmatrix} \tag{31}$$

Now, two subsystems can be formed, as indicated by the lines drawn in the matrices. The expanded system can be considered as an interconnection of two disjoint subsystems,

$$\left[\begin{array}{cc|cc} E_{11} & E_{12} & 0 & 0 \\ E_{21} & E_{22} & 0 & 0 \\ \hline 0 & 0 & E_{22} & E_{23} \\ 0 & 0 & E_{32} & E_{33} \end{array}\right] \begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \hline \dot{z}_2 \\ \dot{z}_3 \end{Bmatrix} = \left[\begin{array}{cc|cc} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 \\ \hline 0 & 0 & A_{22} & A_{23} \\ 0 & 0 & A_{32} & A_{33} \end{array}\right] \times \begin{Bmatrix} z_1 \\ z_2 \\ \hline z_2 \\ z_3 \end{Bmatrix} + \left[\begin{array}{c|c} B_{11} & 0 \\ B_{21} & 0 \\ \hline 0 & B_{22} \\ 0 & B_{32} \end{array}\right] \begin{Bmatrix} u_1 \\ \hline u_2 \end{Bmatrix} \tag{32}$$

and the interconnection terms are

$$E_o\dot{z} = \left[\begin{array}{cc|cc} 0 & 0 & 0 & E_{13} \\ 0 & 0 & 0 & E_{23} \\ \hline E_{21} & 0 & 0 & 0 \\ E_{31} & 0 & 0 & 0 \end{array}\right] \begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \hline \dot{z}_2 \\ \dot{z}_3 \end{Bmatrix}, \qquad \text{etc.}$$

The expansion and the properties of the expanded system are governed by the inclusion principle, which is defined for generalized state-space systems below.

Consider two dynamic systems, Σ [Eqs. (25)] and $\tilde{\Sigma}$:

$$\tilde{\Sigma}: \begin{cases} \tilde{E}\dot{\tilde{z}}(t) = \tilde{A}\tilde{z}(t) + \tilde{B}u(t) \qquad E\tilde{z}(0) = E\tilde{z}_0 \\ y(t) = \tilde{C}\tilde{z}(t) \end{cases} \tag{33}$$

where only the state space is expanded (i.e., $\dim\tilde{z} > \dim z$) and the dimensions of the input and output vectors of the two systems are the same.

Definition 4.1. System $\tilde{\Sigma}$ includes system Σ if there exists a pair of constant matrices (U, V) such that $UV = I$ and, for any admissible pair $(z_0, u(t))$, a $\tilde{z}(t)$ exists for system $\tilde{\Sigma}$, and

$$z(t; z_0, u) = U\tilde{z}(t; Vz_0, u), \qquad y[z(t)] = y[\tilde{z}(t)] \qquad t \geq 0$$

This definition is similar to the definition of inclusion found in Ref. 2, except that the systems are described by descriptor models rather than standard state-space models. System $\tilde{\Sigma}$ is called an expansion of system Σ, and conversely, Σ is called a contraction of $\bar{\Sigma}$. The expanded system matrices can be created with the transformations

$$\bar{E} = VEU + E_c, \qquad \bar{A} = VAU + A_c$$
$$\bar{B} = VB + B_c, \qquad \bar{C} = CU + C_c \tag{34}$$

Since U and V are not square matrices, these are singular transformations. Consequently, the characteristics of the expanded system are not unique, and one can choose the complementary matrices E_c, A_c, B_c, and C_c.

Two specific transformations that are frequently used create expansions so that Σ is a restriction of $\bar{\Sigma}$ or so that Σ is an aggregation of $\bar{\Sigma}$. Both of these transformations are mathematical constructs with no physical implications in general, although in some cases it may be possible to attach specific meanings to them. As such, they are applicable to any dynamic system.

The first type of expansion uses a monic V (i.e., V has full column rank). If, for any admissible pair $(z_0, u(t))$,

$$\bar{z}(t; Vz_0, u) = Vz(t; z_0, u), \qquad y[z(t)] = y[\bar{z}(t)] \qquad t \geq 0$$

holds, then system Σ is called a restriction of $\bar{\Sigma}$. Restriction completely specifies the state trajectories of $\bar{\Sigma}$ for a limited set of initial conditions given by $\bar{z}_0 = Vz_0$.

The second type of expansion is obtained when the matrix U is specified. For an epic (full-row-rank) matrix U, if

$$z(t; U\bar{z}_0, u) = U\bar{z}(t; \bar{z}_0, u), \qquad y[z(t)] = y[\bar{z}(t)] \qquad t \geq 0$$

hold for any initial states $\bar{z}_0$ and input $u(t)$, for which $\bar{z}(t)$ exists, then Σ is called an aggregation of $\bar{\Sigma}$. Aggregation defines the projection of the state trajectories of $\bar{\Sigma}$ onto the state space of Σ for arbitrary initial conditions in $\bar{\Sigma}$. The following theorem establishes conditions for aggregation and restriction for generalized state-space systems.

Theorem 4.1. System Σ is a restriction of system $\bar{\Sigma}$ if $|s\bar{E} - \bar{A}| \neq 0$ identically and

$$E_cV = 0, \qquad A_cV = 0, \qquad B_c = 0, \qquad C_cV = 0$$

or equivalently,

$$\bar{E}V = VE, \qquad \bar{A}V = VA, \qquad \bar{B} = VB, \qquad \bar{C}V = C$$

Similarly, Σ is an aggregation of $\bar{\Sigma}$ if $|s\bar{E} - \bar{A}| \neq 0$ identically and

$$UE_c = 0, \qquad UA_c = 0, \qquad UB_c = 0, \qquad C_c = 0$$

or equivalently,

$$U\bar{E} = EU, \qquad U\bar{A} = AU, \qquad U\bar{B} = B, \qquad \bar{C} = CU$$

Proof. Laplace transform Σ and $\bar{\Sigma}$ to obtain

$$sEz(s) = Az(s) + Bu(s) + Ez_0, \qquad y(s) = Cz(s) \tag{35}$$

$$s\bar{E}\bar{z}(s) = \bar{A}\bar{z}(s) + \bar{B}u(s) + \bar{E}\bar{z}_0, \qquad y(s) = \bar{C}\bar{z}(s) \tag{36}$$

For the case of restriction, premultiply Eq. (35a) by V and use $\bar{E}V = VE$, $\bar{A}V = VA$, and $\bar{B} = VB$ to obtain

$$s\bar{E}Vz = \bar{A}Vz(s) + \bar{B}u(s) + \bar{E}Vz_0 \tag{37}$$

which implies that $\bar{z}(s) = Vz(s)$. Uniqueness of $\bar{z}(s)$ comes from the condition that $|s\bar{E} - \bar{A}| \neq 0$ identically. The requirement that $y = Cz(s) = (CU + C_c)\bar{z}(s)$ is the same as $C_c\bar{z} = C_cVz = 0$, so $C_cV = 0$.

Next, consider aggregation. For any pair $(u(t), \bar{z}_0)$ admissible in $\bar{\Sigma}$, $\bar{z}(s)$ exists, so we can premultiply Eq. (36a) by U and use $U\bar{E} = EU$, $U\bar{A} = AU$, $U\bar{B} = B$, which yields Eq. (35a) when $U\bar{z}(s) = z(s)$ and the pair $(u(t), U\bar{z}_0)$ is admissible in Σ. The requirement that $C_c = 0$ is satisfied directly by $y = Cz(s) = (CU + C_c)\bar{z}(s)$. QED

The complementary matrices are not unique, so they can be chosen so that the expanded system has a desired structure. Once the expanded system has been created, standard disjoint decomposition methods can be applied to the interconnected system.

The inclusion principle is also applicable to performance indices and estimators associated with the dynamic systems. For an LQR, performance indices J and $\bar{J}$ are associated with systems Σ and $\bar{\Sigma}$ respectively, where

$$J = \frac{1}{2}\int_0^\infty (z^TQz + u^TRu)\,dt$$
$$\bar{J} = \frac{1}{2}\int_0^\infty (\bar{z}^T\bar{Q}\bar{z} + u^TRu)\,dt \tag{38}$$

If the matrix $\bar{Q}$ is chosen so that $Q = V^T\bar{Q}V$ and the expansion of the dynamic system satisfies the restriction conditions, then any state feedback gain matrix $\bar{G}$ designed in the expanded space is contractible to the original space, giving $G = \bar{G}V$ and $J = \bar{J}$.

Normally, $\bar{G}$ is computed for the disjoint subsystems in the expanded space. That is, we solve the following problem:

Minimize $\bar{J}$

Subject to

$$E_d\dot{\bar{z}} = A_d\bar{z} + B_du$$

When a structural dynamics system is expanded so that the decoupled systems are its substructures, this problem will be exactly the SCS LQ problem (problem 2) discussed in the previous section.

For estimators of Σ and $\bar{\Sigma}$, the same expansions used for the dynamic systems are used. Let $\mathcal{E}$ and $\bar{\mathcal{E}}$,

$$\mathcal{E}\!: E\dot{q} = Aq + Bu + F(y - Cq)$$
$$\bar{\mathcal{E}}\!: \bar{E}\dot{\bar{q}} = \bar{A}\bar{q} + \bar{B}u + \bar{F}(y - \bar{C}\bar{q}) \tag{39}$$

be estimators for Σ and $\bar{\Sigma}$, respectively. When $\mathcal{E}$ is expanded so that it is an aggregation of $\bar{\mathcal{E}}$, any estimator gain matrix designed in the expanded space is contractible to the original space. If $\bar{F}$ is the estimator gain matrix designed in the expanded space, the contracted gain matrix is given by

$$F = U\bar{F} \tag{40}$$

Here we must note that the most significant difference between overlapping decompositions of standard state-space systems and descriptor systems is in the contracted estimator gain matrices. A simple way to illustrate this is to assume that E^{-1} exists so we can convert $\mathcal{E}$ to standard form,

$$\dot{\hat{z}} = E^{-1}A\hat{z} + E^{-1}Bu + L(y - C\hat{z}), \qquad L = E^{-1}F \tag{41}$$

If we compute L based on an overlapping decomposition of Eq. (41), then using the same notation as above, we get $L = U\bar{L}$. By premultiplying Eq. (39a) by E^{-1} after obtaining F from $\bar{F}$, computed in the expanded space, we get $L = E^{-1}U\bar{F}$ and $E^{-1}U\bar{F} \neq U\bar{L}$. Essentially, the difference depends on premultiplying by E^{-1} before or after the expansion and contraction steps. It can be easily seen that the only fully populated rows in $U\bar{L}$ are those corresponding to the overlapped state estimates, whereas in $E^{-1}U\bar{F}$, all of the rows will be fully populated unless E is diagonal.

V. Substructuring Decomposition and Inclusion Principle

The substructuring decomposition introduced in Sec. II may be interpreted from a purely mathematical point of view as a special case of an overlapping decomposition of generalized state-space systems. When the state coupling matrix $\bar{T}$ introduced in Eq. (10) is chosen to be one of the expansion matrices used to create an expanded system, the substructures can be broken out as the decoupled subsystems in the expanded space. Consider again the two-component structure shown in Fig. 1. Let the nodes of the structure be partitioned into interior nodes and boundary nodes, as discussed in Sec. II. The states can be chosen as the generalized displacements and velocities and organized as

$$\left\{\begin{array}{c} z_\alpha^i \\ \hline z^b \\ \hline z_\beta^i \end{array}\right\} = \left\{\begin{array}{c} x_\alpha^i \\ \dot{x}_\alpha^i \\ \hline x^b \\ \dot{x}^b \\ \hline x_\beta^i \\ \dot{x}_\beta^i \end{array}\right\}$$

The matrices of the generalized state-space equations can be partitioned into

$$E = \begin{bmatrix} E_\alpha^{ii} & E_\alpha^{ib} & 0 \\ E_\alpha^{bi} & E_\alpha^{bb} + E_\beta^{bb} & E_\beta^{bi} \\ 0 & E_\beta^{ib} & E_\beta^{ii} \end{bmatrix}$$

$$A = \begin{bmatrix} A_\alpha^{ii} & A_\alpha^{ib} & 0 \\ A_\alpha^{bi} & A_\alpha^{bb} + A_\beta^{bb} & A_\beta^{bi} \\ 0 & A_\beta^{ib} & A_\beta^{ii} \end{bmatrix}$$

$$B = \begin{bmatrix} B_\alpha^i & 0 \\ 0 & 0 \\ 0 & B_\beta^i \end{bmatrix}, \qquad \Gamma = \begin{bmatrix} \Gamma_\alpha^i & 0 \\ 0 & 0 \\ 0 & \Gamma_\beta^i \end{bmatrix}$$

$$C = \begin{bmatrix} C_\alpha & 0 & 0 \\ 0 & 0 & C_\beta \end{bmatrix}$$

to correspond to the partitioning of the state vector, and the above submatrices are related to the structural dynamics system matrices of Eq. (7) by the following:

$$E_\alpha^{ii} = \begin{bmatrix} D_\alpha^{ii} & M_\alpha^{ii} \\ M_\alpha^{ii} & 0 \end{bmatrix}$$

$$E_\alpha^{bb} + E_\beta^{bb} = \begin{bmatrix} D_\alpha^{bb} + D_\beta^{bb} & M_\alpha^{bb} + M_\beta^{bb} \\ M_\alpha^{bb} + M_\beta^{bb} & 0 \end{bmatrix} \qquad \text{etc.}$$

$$A_\alpha^{ii} = \begin{bmatrix} -K_\alpha^{ii} & 0 \\ 0 & M_\alpha^{ii} \end{bmatrix}$$

$$A_\alpha^{bb} + A_\beta^{bb} = \begin{bmatrix} -\left(K_\alpha^{bb} + K_\beta^{bb}\right) & 0 \\ 0 & M_\alpha^{bb} + M_\beta^{bb} \end{bmatrix} \qquad \text{etc.}$$

$$B_\alpha^i = \begin{bmatrix} P_\alpha^i \\ 0 \end{bmatrix}, \qquad B_\beta^i = \begin{bmatrix} P_\beta^i \\ 0 \end{bmatrix}, \qquad \text{etc.}$$

Notice that the matrices B, Γ, and C reflect the local input and local output conditions in that the rows (or columns) associated with boundary states have only zero entries.

The model of the complete structure can be expanded so that it represents a restriction of the expanded system by letting

$$U = \tilde{T}^I, \qquad V = \tilde{T} \tag{42}$$

where $\tilde{T}^I$ is the pseudoinverse of $\tilde{T}$. If the complimentary matrices are chosen as

$$E_c = \begin{bmatrix} 0 & \frac{1}{2}E_\alpha^{ib} & -\frac{1}{2}E_\alpha^{ib} & 0 \\ 0 & \frac{1}{2}\left(E_\alpha^{bb} + E_\beta^{bb}\right) & -\frac{1}{2}\left(E_\alpha^{bb} + E_\beta^{bb}\right) & 0 \\ 0 & -\frac{1}{2}\left(E_\alpha^{bb} + E_\beta^{bb}\right) & \frac{1}{2}\left(E_\alpha^{bb} + E_\beta^{bb}\right) & 0 \\ 0 & -\frac{1}{2}E_\beta^{ib} & \frac{1}{2}E_\beta^{ib} & 0 \end{bmatrix}$$

$$A_c = \begin{bmatrix} 0 & \frac{1}{2}A_\alpha^{ib} & -\frac{1}{2}A_\alpha^{ib} & 0 \\ 0 & \frac{1}{2}\left(A_\alpha^{bb} + A_\beta^{bb}\right) & -\frac{1}{2}\left(A_\alpha^{bb} + A_\beta^{bb}\right) & 0 \\ 0 & -\frac{1}{2}\left(A_\alpha^{bb} + A_\beta^{bb}\right) & \frac{1}{2}\left(A_\alpha^{bb} + A_\beta^{bb}\right) & 0 \\ 0 & -\frac{1}{2}A_\beta^{ib} & \frac{1}{2}A_\beta^{ib} & 0 \end{bmatrix}$$

$$B_c = 0, \qquad \Gamma_c = 0, \qquad C_c = 0$$

then the expanded matrices are

$$\bar{E} = \left[\begin{array}{cc|cc} E_\alpha^{ii} & E_\alpha^{ib} & 0 & 0 \\ E_\alpha^{bi} & E_\alpha^{bb} + E_\beta^{bb} & 0 & E_\beta^{bi} \\ \hline E_\alpha^{bi} & 0 & E_\alpha^{bb} + E_\beta^{bb} & E_\beta^{bi} \\ 0 & 0 & E_\beta^{ib} & E_\beta^{ii} \end{array}\right]$$

$$\bar{A} = \left[\begin{array}{cc|cc} A_\alpha^{ii} & A_\alpha^{ib} & 0 & 0 \\ A_\alpha^{bi} & A_\alpha^{bb} + A_\beta^{bb} & 0 & A_\beta^{bi} \\ \hline A_\alpha^{bi} & 0 & A_\alpha^{bb} + A_\beta^{bb} & A_\beta^{bi} \\ 0 & 0 & A_\beta^{ib} & A_\beta^{ii} \end{array}\right]$$

$$\bar{B} = \left[\begin{array}{c|c} B_\alpha^i & 0 \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & B_\beta^i \end{array}\right], \qquad \bar{\Gamma} = \left[\begin{array}{c|c} \Gamma_\alpha^i & 0 \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & \Gamma_\beta^i \end{array}\right]$$

$$\bar{C} = \left[\begin{array}{cc|cc} C_\alpha & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & C_\beta \end{array}\right]$$

Now we can consider a disjoint decomposition of the expanded model of the structure. When the interconnection matrices are chosen as

$$E_{11} = \begin{bmatrix} 0 & 0 \\ 0 & E_\beta^{bb} \end{bmatrix}, \qquad E_{12} = \begin{bmatrix} 0 & 0 \\ 0 & E_\beta^{bi} \end{bmatrix}$$

$$E_{21} = \begin{bmatrix} E_\alpha^{bi} & 0 \\ 0 & 0 \end{bmatrix}, \qquad E_{22} = \begin{bmatrix} E_\alpha^{bb} & 0 \\ 0 & 0 \end{bmatrix}$$

$$A_{11} = \begin{bmatrix} 0 & 0 \\ 0 & A_\beta^{bb} \end{bmatrix}, \qquad A_{12} = \begin{bmatrix} 0 & 0 \\ 0 & A_\beta^{bi} \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} A_\alpha^{bi} & 0 \\ 0 & 0 \end{bmatrix}, \qquad A_{22} = \begin{bmatrix} A_\alpha^{bb} & 0 \\ 0 & 0 \end{bmatrix}$$

the decoupled subsystems are the individual substructures. Now substructure gain matrices can be calculated and contracted to give a global gain matrix. We see that $\bar{G} = \tilde{G}^\circ$ and $G = G^\oplus$, and since $V = \tilde{T}$, contraction and SCS assembly produce the same global gain matrix.

In a similar fashion, the assembled structure model may also be expanded so that it represents an aggregation of the expanded system by letting

$$U = \tilde{T}^T, \qquad V = \tilde{T}^{T^I} \tag{43}$$

and choosing the complimentary matrices appropriately. The decoupled subsystems in the expanded space can be the substructures and decentralized estimators can be calculated. The estimator gain matrices are contracted to form a global estimator gain matrix. Again, assembly and contraction are equivalent. Hence, we see that substructuring decomposition and the SCS method of controller design can be interpreted as a decentralized control approach using overlapping subsystems. Of course, it is more convenient to interpret the SCS method in its original form since there is no need to explicitly expand and contract the structure model.

VI. Example

A plane truss structure (Fig. 2) is used to demonstrate the applicability of the SCS method. Although it is relatively simple, this structure has some properties that make it a good example. First, the α component is cantilevered and the β substructure is free–free. This means that a subcontroller must deal with the rigid-body modes of the β substructure, which are not present in the assembled structure. Second, the assembled structure has a mode well below the lowest

Table 1 Sensor and actuator locations on example truss structure

Output signal	Node	Actuator type	Node
X velocity	5, 11	X force	6, 13
Y velocity	6, 12	Y force	5, 10, 14
X position	14		
Y position	12, 13		

Table 2 Open-loop frequencies of substructure and assembled structure (below 110 rad/s)

α substructure		β substructure		Assembled structure	
Mode	Frequency, rad/s	Mode	Frequency, rad/s	Mode	Frequency, rad/s
1	6.34	1	0	1	2.22
2	29.21	2	0	2	11.96
3	66.66	3	0	3	29.04
4	101.7	4	58.38	4	49.39
5	109.7	5	102.6	5	60.46
				6	71.54
				7	94.94
				8	109.0

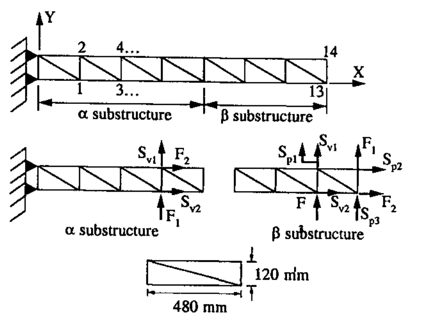

Fig. 2 Truss example structure.

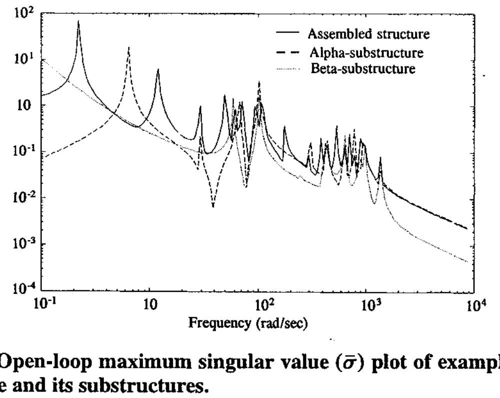

Fig. 3 Open-loop maximum singular value ($\bar{\sigma}$) plot of example truss structure and its substructures.

modes of the individual substructures (see Fig. 3 and Table 1). Finally, a cantilevered truss is a typical CSI structure. For example, the NASA Langley Minimast structure is a cantilevered truss structure, as are other test structures.[10,14]

The example truss is assumed to be made of aluminum tubes with outer diameter $d_o = 4$ mm and thickness $t = 1$ mm. The dimensions of each bay are shown in Fig. 2. Five force-type actuators are used to control the structure, as shown in Fig. 2 and listed in Table 2. Two actuators are associated with the α substructure and three actuators are associated with the β substructure. The output signals are velocity and position. The α substructure has two velocity outputs, and the β substructure has two velocity outputs and three position outputs, which are also shown in Fig. 2 and listed in Table 2.

Control design requires that the systems be completely controllable and completely observable from the minimized outputs and from the measurements. These requirements mean that the β substructure must have at least three position measurements and three actuators to fully sense and control the three rigid-body modes. Since the α substructure is cantilevered, there are no restrictions on the number or type of sensors and actuators.

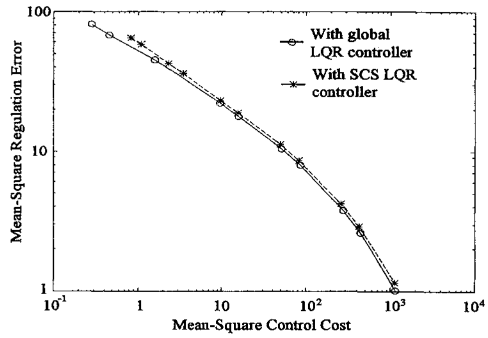

Fig. 4 Comparison of optimal and SCS stochastic regulators ($W = 1$).

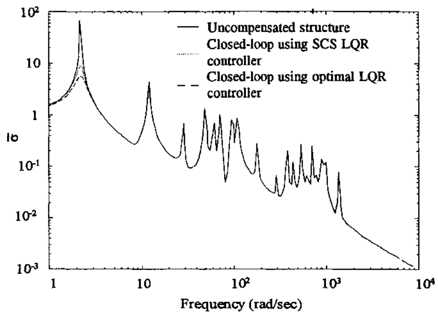

Fig. 5 Closed-loop LQR $\bar{\sigma}$ plot of assembled structure, $\rho^{scs} = 100$.

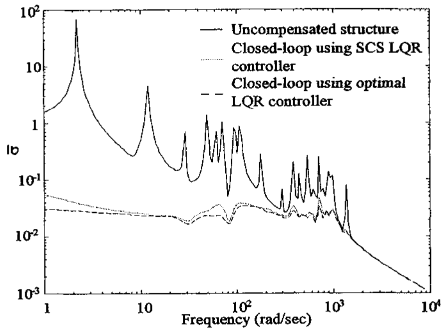

Fig. 6 Closed-loop LQR $\bar{\sigma}$ plot of assembled structure, $\rho^{scs} = 0.001$.

The performance indices used in the SCS regulator designs were chosen to be

$$J_r = \frac{1}{2}\int_0^\infty \left[\dot{x}_r^T M_r \dot{x}_r + x_r^T K_r x_r + 0.1 y_{r_p}^T y_{r_p} + \rho u_r^T u_r\right] \mathrm{d}t \qquad r = \alpha, \beta \quad (44)$$

The third term is the square of the substructure position measurements. This term is included primarily to extend additional control authority to the rigid-body modes of the β substructure. It is straightforward to show that these performance indices are assemblable and satisfy the inclusion conditions defined earlier.

The simulation results are shown in Figs. 4–7. Figure 4 compares the performance of the optimal regulator and the SCS regulator by plotting the mean-square regulation error vs the mean-square control cost, assuming a unit intensity stochastic disturbance and using control penalty factors ranging from $0.001 \leq \rho \leq 100$. The bounds on ρ were chosen on the basis of the closed-loop response. (Incidentally, the regulators were stable even for smaller values of ρ.) Above $\rho = 100$, the controller exerted very little influence (Fig. 5), and at $\rho = 0.001$ the controller was very aggressive (Fig. 6). The LQR results are very good. The SCS full-state feedback controller performance compares very well with the optimal full-state feedback controller over the complete range of ρ values.

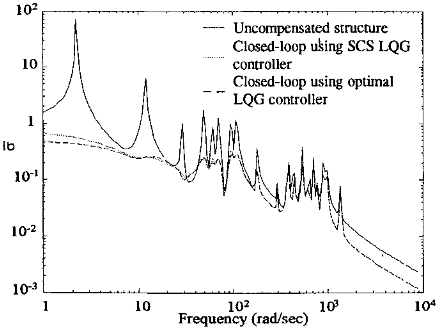

Fig. 7 Closed-loop LQG $\bar{\sigma}$ plot of assembled structure, $\rho^{scs} = 0.1$, $V/W = 1 \times 10^{-3}$.

In the dynamic output feedback case, the estimator was a Kalman filter. Estimator gains were calculated for each substructure for noise intensity ratios $\hat{V}/\hat{W} \geq 1 \times 10^{-4}$. The SCS LQG controller was stable over the whole range of $\hat{V}/\hat{W}$. Figure 7 shows the closed-loop maximum singular-value ($\bar{\sigma}$) plot (i.e., a frequency response plot) of the complete structure using the SCS LQG controller and the global optimal LQG controller designed with $\rho = 0.1$ and $\hat{V}/\hat{W} = 1 \times 10^{-3}$. Again, the results are excellent; the frequency response curves almost coincide. Increasing $\hat{V}/\hat{W}$ reduced the effectiveness of the controllers, as expected, and again the SCS LQG controller compared well with the optimal one.

VII. Summary

A decentralized LQ control design method called the SCS method has been presented for control design of flexible structures. The basic idea is to design controllers for the substructures of a structure and then to create a global controller by assembling the subcontrollers using the same coupling process employed to assemble the substructures. If all substructure states are available for feedback, then the resulting controller is decentralized. When estimators are used, a global controller results.

The SCS method is attractive because it is a decentralized design approach in which no explicit decomposition of the structure model is necessary, since the substructure models often already exist. The SCS controller is very adaptable because the subcontrollers are designed independently, and therefore the controller can be easily updated if part of the structure changes. In addition, since it is a decentralized method, the computational cost of designing the controller is reduced.

It was shown that this method is closely related to decentralized controller design using overlapping subsystems. In fact, the SCS method can be interpreted as a specific case of decentralized controller design using overlapping subsystems expressed in generalized form. This was demonstrated by extending the concepts of overlapping decompositions and the inclusion principle to generalized state-space systems. However, it is more convenient to interpret the SCS method in its original form because it bypasses the need to explicitly expand and decompose the structure model.

The feasibility of the SCS method was numerically demonstrated for a plane truss structure. The SCS controller compared well with a global optimal controller. Further research in substructure-based structural control includes developing techniques to assess robustness to modeling errors and to disturbances introduced from the assembly of the substructures.

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